Trigonometry and Complex Numbers

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1 Introduction

This document is a short introduction to the relation between complex numbers and trigonometry, and shows how to approach trigonometrical problems using complex numbers. This is not completely complete, maybe I'll add something else later. The targets of this document are beginner Olympiad problem solvers who have begun studying complex numbers. It is assumed that the reader has the basic knowledge of trigonometry (definitions, periods, addition formulas and related identities).

2 Basic Facts

We define the complex numbers $\mathbb{C}$ as

$$\mathbb{C} = \left\{ a + bi \mid (a, b) \in \mathbb{R}^2 \right\}, \quad i^2 = -1.$$

In a complex number $z = a + bi$, a is called the real part and written Re $z = a$, while b is called the imaginary part and written Im $z = b$. One good thing about dealing with complex numbers in equations is that one can equate the real and imaginary parts. For some real parameters A, B, C, D we have

$$A + Bi = C + Di \;\Leftrightarrow\; A = C,\; B = D.$$

Another helpful thing is the linearity of the Re or Im

$$\text{Re} \sum = \sum \text{Re}\ , \qquad \text{Im} \sum = \sum \text{Im}.$$

3 Euler's Formula

The following formula was given by Leonhard Euler and is a very useful one relating complex numbers and trigonometry. For any complex x we have

$$e^{ix} = \cos x + i \sin x, \qquad \left[e = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n \approx 2.7183. \right]$$

There are many ways to prove this. One outline is included here: first, rewrite the equation as

$$\frac{\cos x + i \sin x}{e^{ix}} = 1,$$

consider the function $y = e^{-ix}(\cos x + i \sin x)$ and differentiate it to get 0. So the function must be constant. Substitute $x = 0$ to get $y = 1$, hence $y = 1$ for all x. Note that setting $x = \pi$ we get a beautiful identity $e^{i\pi} + 1 = 0$.

4 De Moivre's Formula

De Moivre's Formula helps ease out computations a lot. It states that for any complex x and integer n

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx.$$

The proof is one liner if we apply Euler's formula twice

$$(\cos x + i \sin x)^n = \left(e^{ix}\right)^n = e^{i(nx)} = \cos nx + i \sin nx.$$

Notice that if $z = e^{ix} = \cos x + i \sin x$ is a complex number, then Re $z = \cos x$ and Im $z = \sin x$.

5 Complex Approaches to Trig Problems!

Example 1. Derive the addition formulas of sine and cosine.

Solution. Just notice that by applications of Euler's formula

$$\begin{aligned} \cos(x+y) + i\sin(x+y) &= e^{i(x+y)} \\ &= e^{ix} \cdot e^{iy} \\ &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y) \end{aligned}$$

Now equating real and imaginary part of both sides, we have

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

deriving the formulas. □

Example 2. Prove that

$$\sum_{k=0}^{n-1} \cos \frac{2k\pi}{n} = 0 \qquad \forall\ n \in \mathbb{N} - \{1\}.$$

Solution. Let $z = e^{2\pi i/n} = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Then $z^n = e^{2\pi i} = 1 \implies z^n - 1 = 0$. Since $z \neq 1$

$$z^n - 1 = (z-1)\sum_{k=0}^{n-1} z^k = 0 \implies \sum_{k=0}^{n-1} z^k = 0 \implies \text{Re}\sum_{k=0}^{n-1} z^k = 0.$$

Now by application of De Moivre's theorem

$$z^k = \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}\right)^k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \implies \text{Re } z^k = \cos \frac{2k\pi}{n}.$$

Therefore by linearity of Re we finally have

$$0 = \text{Re}\sum_{k=0}^{n-1} z^k = \sum_{k=0}^{n-1} \text{Re } z^k = \sum_{k=0}^{n-1} \cos \frac{2k\pi}{n}. \ \square$$

Example 3. Show the following (take it granted that the sum converges absolutely)

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n} = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}.$$

Solution. Consider $e^{in\theta} = \cos n\theta + i \sin n\theta$, so that Re $e^{in\theta} = \cos n\theta$. Hence by linearity of Re

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n} = \sum_{n=0}^{\infty} \frac{\text{Re } e^{in\theta}}{2^n} = \text{Re} \sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2} \right)^n$$

which is an infinite geometric series sum. Evaluating we have

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n} = \text{Re} \sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2} \right)^n = \text{Re } \frac{1}{1 - e^{i\theta}/2} = \text{Re } \frac{2}{2 - e^{i\theta}}.$$

Finally use the identity Re $z = (z + \overline{z})/2$, replace $e^{i\theta}$ using Euler's formula and simplify to get

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n} = \text{Re } \frac{2}{2 - e^{i\theta}} = \frac{1}{2} \left(\frac{2}{2 - e^{i\theta}} + \frac{2}{2 - e^{-i\theta}} \right) = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}$$

as desired. □

6 Problems to Try

Problem 1. (IMO 1963/5) Show that

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$$

Problem 2. (Proofathon) Show that for all odd $k \geq 3$

$$\cos\frac{\pi}{k} + \cos\frac{3\pi}{k} + \cdots + \cos\frac{(k-2)\pi}{k} = \frac{1}{2}.$$

(Can you see how this is a generalization to **Problem 1**?)

Problem 3. Show the following (take it granted that the sum is absolutely convergent)

$$\sum_{n=0}^{\infty} \frac{\cos^2 n\theta}{2^n} = \frac{7 - 5\cos 2\theta}{5 - 4\cos 2\theta}.$$

(You may use the result proven in **Example 3**.)